$C(1,)$ AS M-C ON $\{3,4\}$ IS $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

P^{400}_{63} IS EXACTLY

$P(T_3 \le 400, T_3 \text{ IS EVEN} / X_0 = 6)$

$= 1/6 \left[1/6 + (1/6)^3 \ldots + (1/6)^{395} \right]$

$\approx \frac{1}{36} \frac{1}{1 - 1/36} = \boxed{\frac{1}{35}}$

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/6 & 0 & 1/6 & 0 & 1/6 \\ 1/3 & 1/3 & 0 & 1/6 & 0 & 0 & 1/6 \end{pmatrix}$$

a) What are the transient states of the chain, what are the communication classes?

b) If $X_0 = 6$, what is the probability that the chain will visit state 6 exactly 3 times?

c) Approximately what is P^{400}_{62}? What approximately is P^{400}_{63}?

a) THE COMMUNICATING CLASSES ARE

$\{0,1,2\}$ $\{3,4\}$ $\{5\}$ $\{6\}$

$\{5\}$ IS A SINGLE CLASS AS ITS ROW HAS ALL ZEROS

$\{6\}$ IS A SINGLE CLASS AS $i \to 6$ ONLY FOR 5, 6 AND $6 \not\to 5$

$\{3,4\}$ IS A CLOSED COMMUNICATING CLASS AS $P_{34}=1$ $P_{43}=1$

$\{0,1,2\}$ IS A CLASS AS $0 \to 1$ $0 \to 2$ $2 \to 0$ $1 \to 0$ $(P^2_{10} > 0.)$

(ii) $\{5\}$ $\{6\}$ ARE TRANSIENT.

b) VISITED IS AMBIGUOUS BY MARKOV PROPERTY IT IS $(1/6)^2\, 5/6$ IF $X_0=6$ IS COUNTED AS A VISIT OTHERWISE $(1/6)^3 5/6$.

c) CONSIDER MC ON $\{0,1,2\}$ THE CLOSED COMMUNICATING CLASS OF 2. THE STATIONARY DISTRIBUTION ON $\{0,1,2\}$ π SATISFIES $\pi \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \pi$

SO $\pi(0) = 1/2\, \pi(0) + 1/2\, \pi(2)$

$\Rightarrow \pi(0) = \pi(2)$

$\pi(2) = \frac{1}{4}\pi(0) + \pi(1) = \frac{1}{4}\pi(2) + \pi(1)$

$\Rightarrow \frac{3}{4}\pi(2) = \pi(1)$

$\Rightarrow 1 = \pi(0) + \pi(1) + \pi(2)$

$= \pi(2)(1 + 3/4 + 1) \Rightarrow \pi(2) = \frac{4}{11}$

$\Rightarrow \boxed{P^{400}_{62} \approx 4/5\; 4/11}$

2) For the Markov chain on $\mathbb{Z}_+$ with

$$P_{n0} = 1 - \tfrac{n^\alpha}{(n+1)^\alpha},\ n \ge 1,\ P_{01} = 1$$
$$P_{n\ n+1} = \tfrac{n^\alpha}{(n+1)^\alpha},$$

is the chain irreducible, ?, aperiodic ? . Show this.

For which values of α is it recurrent ? positive recurrent ? In the case of positive recurrence Give an expression for the equilibrium distribution ?

(i) THE CHAIN IS IRREDUCIBLE AS $\forall n$ $P_{0n}^{n} > 0$ AND $P_{n0} > 0$.
SO $0 \to n$ AND $n \to 0$ SO THERE IS A SINGLE COMMUNICATING CLASS. $P_{00}^2 = P_{01}P_{10} > 0$.

$P_{00}^3 = P_{01}P_{12}P_{20} > 0$ SO THE $\gcd\{n : P_{00}^n > 0\} = 1$

(ii) $P(T_0 > n \mid X_0 = 0) = P_{01}P_{12} \cdots P_{n-1\,n}$

$= 1\, \frac{1^\alpha}{2^\alpha} \cdot \frac{2^\alpha}{3^\alpha} \cdots \frac{(n-1)^\alpha}{n^\alpha} = \frac{1}{n^\alpha} \to 0$ IF $\alpha > 0$

$= 1$ IF $\alpha = 0$

(IF $\alpha < 0$ NO MARKOV CHAIN)

SO IT IS TRANSIENT IF $\alpha = 0$, RECURRENT IF $\alpha > 0$. IF $\alpha > 0$

$P(T_0 = n \mid X_0 = 0) = P_{01}P_{12} \cdots P_{n-2\,n-1}P_{n-1\,0}$

$= 1\, \frac{1^\alpha}{2^\alpha}\, \frac{2^\alpha}{3^\alpha} \cdots \frac{(n-2)^\alpha}{(n-1)^\alpha} \left(1 - \frac{(n-1)^\alpha}{n^\alpha}\right)$

$= \frac{1}{(n-1)^\alpha}\left(1 - \left(1 - \frac{1}{n}\right)^\alpha\right) \approx \frac{\alpha}{n(n-1)^\alpha}$.

SO $c \sum \frac{1}{(n-1)^\alpha} < E(T_0 \mid X_0 = 0) < C \sum \frac{1}{(n-1)^\alpha}$.

SO IT IS POSITIVE RECURRENT IF AND ONLY IF $\alpha > 1$

(iii) FOR MC $X_0 = 0$ # OF VISITS TO n BEFORE T_0 IS 1 OR 0

IT IS 1 $\Leftrightarrow$ $T_0 > n$ $= \frac{1}{n^\alpha}$ SO $\frac{\pi(n)}{\pi(0)} = \frac{1}{n^\alpha}$

SO $\pi(n) = \frac{1/n^\alpha}{1 + \sum_{j=1}^{\infty} 1/j^\alpha}$.

3) For the transition matrix on $\{0,1,2\}$ $P = \begin{matrix} 0 & 1/4 & 3/4 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$, is the corresponding chain irreducible, aperiodic? Justify your answers.

For a Markov chain $(X_n)_{n\geq 0}$ $P = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}$, calculate $P(X_3 = 2|X_0 = 0)$.

For a Markov chain $(X_n)_{n\geq 0}$ on $\{0,1,2,3\}$

$$P = \begin{pmatrix} 0 & 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \end{pmatrix},$$ calculate the probability that the chain hits 2 before 3 if $X_0 = 0$. What is the expected number of times 2 is hit before the chain returns to 0 if $X_0 = 0$.

(i) THE CHAIN IS IRREDUCIBLE AS $P_{01}, P_{02} > 0$ SO $0 \to 1, 0 \to 2$ AND $P_{10}, P_{20} = 1 > 0$ SO $P_{10}, P_{20} > 0$ SO $1 \to 0, 2 \to 0$ SO LETS $\forall$ i,j. THE CHAIN SATISFIES $P^n_{00} > 0$ IF AND ONLY IF n IS EVEN SO THE PERIOD IS 2 SO CHAIN IS NOT APERIODIC

(ii) $\{X_3=2, X_0=0\} = \{X_0=0, X_1=2, X_2=2, X_3=2\}$
$\cup \{X_0=0, X_1=1, X_2=0, X_3=2\}$

SO $P(X_3=2 | X_0=0) = P_{02}P_{22}P_{22} + P_{01}P_{10}P_{02} = \frac{2}{3}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\,\frac{1}{2}\,\frac{2}{3}$

(iii) LET $h(x) = P(\text{HIT 2 BEFORE 3} | X_0 = x)$
(SO $h(2) = 1$ $h(3) = 0$) BY MARKOV PROPERTY
$h(0) = \frac{1}{4}h(1) + 1/2$. $h(1) = 1/2\, h(0) + 1/4$

SO $h(0) = \frac{1}{4}\left[1/2\, h(0) + 1/4\right] + 1/2 = \frac{1}{8}h(0) + \frac{9}{16}$

$\Rightarrow \frac{7}{8}h(0) = \frac{9}{16}$ $h(0) = 9/14$

(SECOND PART) NOTE P^T IS ALSO A TRANSITION MATRIX SO (THE CHAIN IS IRREDUCIBLE) THE UNIQUE STATIONARY LAW IS $(1/4\ 1/4\ 1/4\ 1/4)$ AND SOLUTION $= \frac{\pi(2)}{\pi(0)} = \frac{1/4}{1/4} = 1$

4) A bag initially 10 red balls and 8 blue balls. Then two at random are taken out and random variable X_0 is the number of blue balls removed. In succession (and at random) one of the two removed balls is returned to the bag and replaced by one of the 16 balls in the bag. X_n is the number of blue balls after n such operations. Approximately what is $\frac{1}{81}\sum_{i=20}^{100} X_i$.

Is it true that a Markov chain with a unique stationary distribution is irreducible?

WE HAVE A M-C WITH STATE SPACE $I = \{0,1,2\}$

THE TRANSITION PROBABILITIES ARE

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ \frac{9}{32} & 1/2 & \frac{7}{32} \\ 0 & 5/8 & 3/8 \end{pmatrix}$$

SO WE CALCULATE STATIONARY DISN (THE CHAIN IS IRREDUCIBLE)

$$\pi(0) = \tfrac{1}{2}\pi(0) + \tfrac{9}{32}\pi(1) \Rightarrow \pi(1) = \tfrac{16}{9}\pi(0)$$

$$\pi(2) = \tfrac{7}{32}\pi(1) + \tfrac{3}{8}\pi(2) \Rightarrow \pi(1) = \tfrac{20}{7}\pi(2)$$

SO $$\pi(1) = \frac{1}{1 + 9/16 + 7/20} = \frac{80}{153}$$

$$\pi(0) = \frac{45}{153} \qquad \pi(2) = \frac{28}{153}$$

SO $\frac{1}{81}\sum_{20}^{100} X_i$ IS $\approx$ $\pi(0)0 + \pi(1)1 + \pi(2)2$.

$$= \frac{80+56}{153}$$

(II) IT IS NOT TRUE eg $I = \{0,1\}$ $P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$